Smallest counterexample to the Fulkerson conjecture must be cyclically 5-edge-connected

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joint work with Giuseppe Mazzuoccolo

Fulkerson Conjecture

Theorem (Petersen, 1891)

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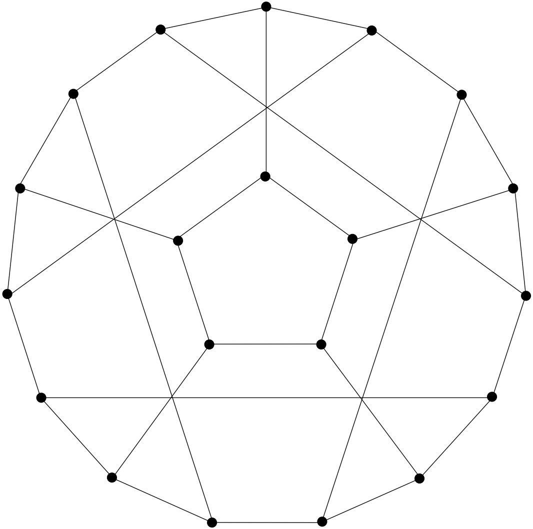
Every bridgeless cubic graphs contains a family of six perfect matchings that together cover each edge exactly twice.

6 perfect matchings on I_5



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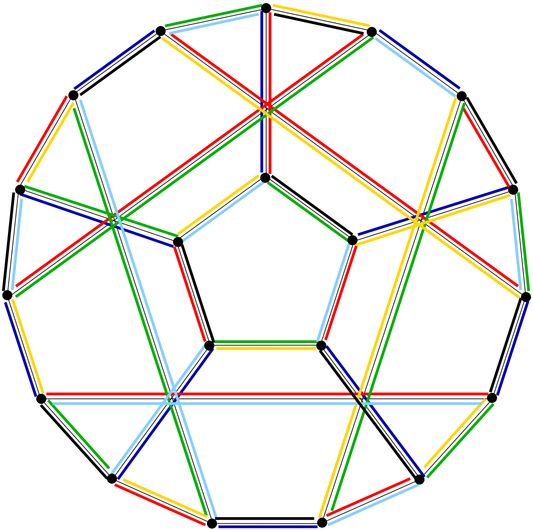

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- DIFFERENT FORMULATION: if we double the edges in a bridgeless cubic graph, we obtain class 1 graph
- if subtraction is permitted, then the constant function 2 can be obtained [Seymour, 1977]

Covering all edges in graph with the same number of perfect matchings

Conjecture (Weak Version of Fulkerson Conjecture)

There exists a constant k such that any bridgeless cubic graphs contains a family of $3k$ perfect matchings that together cover every edge exactly k-times.

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Theorem (Edmonds 1965)

For any bridgeless cubic graph there exists a constant k and $3k$ perfect matchings such that each edge is in k of them.

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- $\exists k \forall G$ $\exists 3k$ PM s.t. every edge is in k PM ... **???** OPEN
- $\forall G \exists k$ $\exists 3k$ PM s.t. every edge is in k PM ... ✓ YES

Results towards the Fulkerson conjecture

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  (i) $M_1 \cup M_2$ induces a 2-regular subgraph of G and
  (ii) the graph obtained from $G \setminus M_i$ by suppressing all degree-2-vertices, is 3-edge-colourable for each i=1,2.

  [Hao, Niu, Wang, Zhang, Zhang, 2009]

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- is true for cubic graphs that are $C_{(8)}$-linked [Hao, Zhang, Zheng, 2018]

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- Fulkerson Conjecture $\Rightarrow$ Berge Conjecture

Theorem (Mazzuoccolo, 2011)

The Berge Conjecture and the Fulkerson Conjecture are equivalent.

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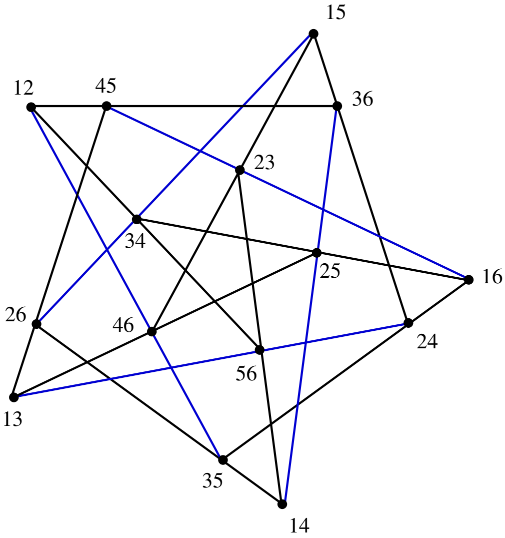
- the Petersen colouring conjecture implies the Fulkerson conjecture

Cremona–Richmond configuration



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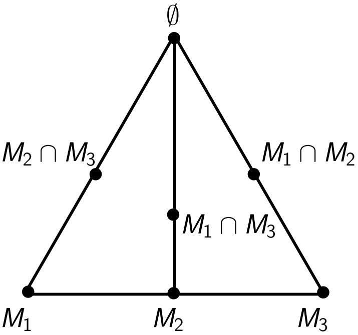
Fan-Raspaud Conjecture

Fan-Raspaud Conjecture, 1994

Every bridgeless cubic graph has three perfect matchings with empty intersection.



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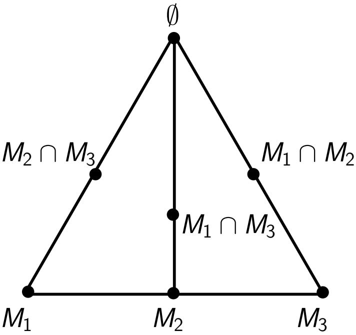
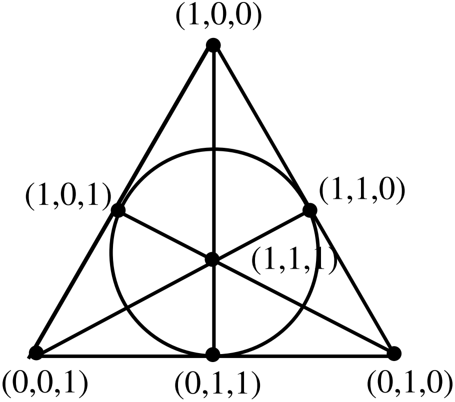


- FC implies Fan-Raspaud conjecture

Fano Plane



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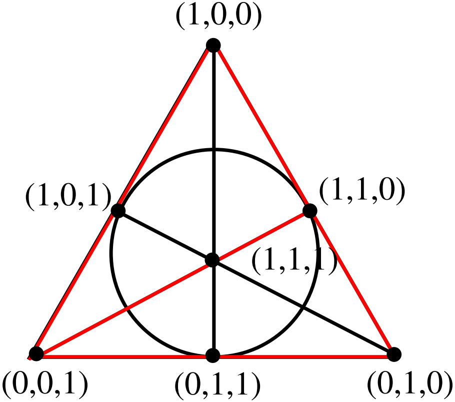

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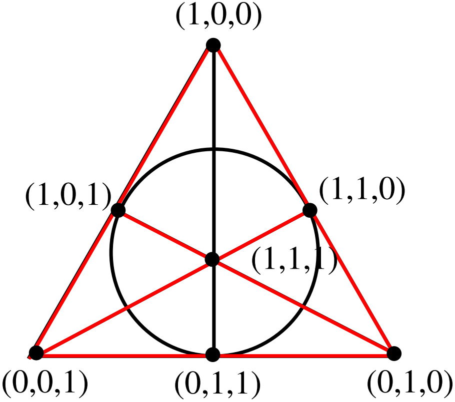

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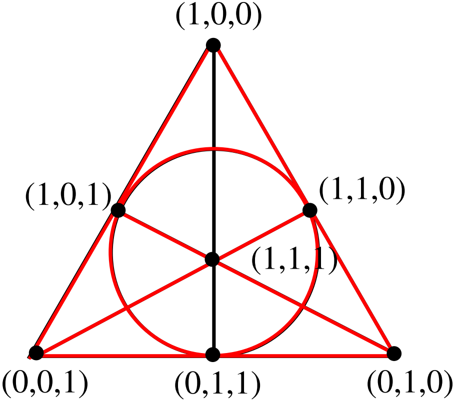
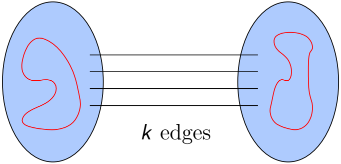


F_6-configuration is bridgeless universal [EM,Škoviera]

Cyclic connectivity and oddness

Cyclic connectivity is the smallest number of edges which have to be removed in order to obtain at least two components containing cycles



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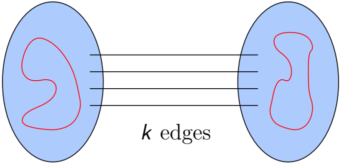

Conjecture (Jaeger, Swart'80)

There are no snarks with cyclic connectivity greater than 6.

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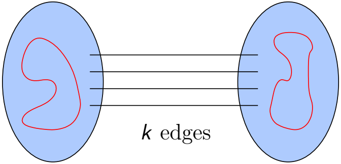

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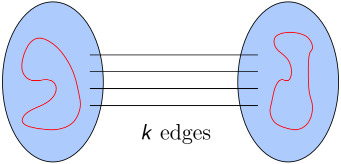

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- $\xi(G) = 0 \Leftrightarrow G$ is 3-edge-colourable

Minimal counterexamples to some conjectures

conj.	girth	cyclic connectivity	oddness
5–flow Conjecture	≥ 11 [Kochol]	≥ 6 [Kochol]	≥ 6 [Mazzuoccollo, Steffen]
5–cycle double cover C.	≥ 12 [Huck]	≥ 4	≥ 6 [Huck]
Fulkerson Conjecture	≥ 5	≥ 4	≥ 2

Reduction of 2- and 3- cycle separating cuts

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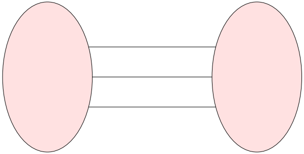

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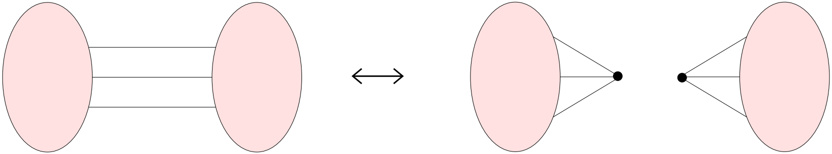

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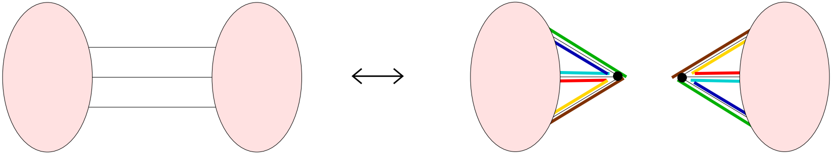

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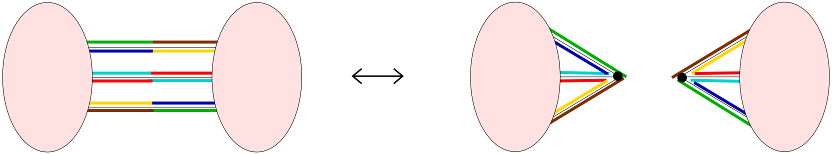

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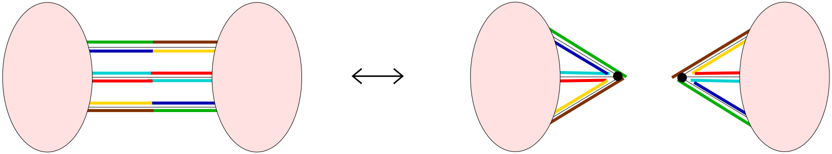

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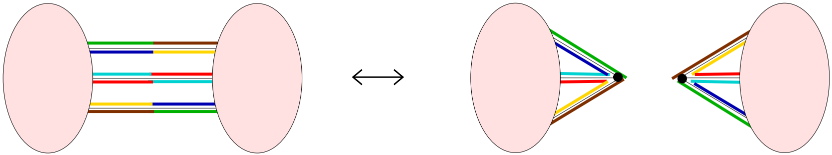

similarly, we can reduce 2-edge-cuts, therefore

Observation

A smallest potential counterexample to the FC is cyclically 4-edge-connected.

Parity lemma

Lemma

Let G be a k-regular multipole. Assume that the edges of G are coloured with k-colours and n_i dangling edges has colour i. Then

$$n_1 \equiv n_2 \equiv \ldots \equiv n_k \pmod 2.$$

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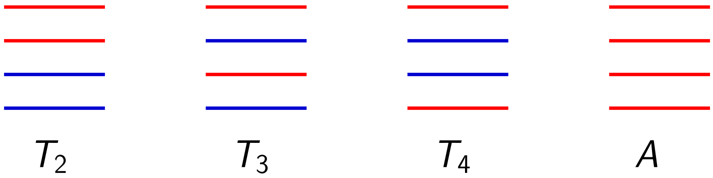

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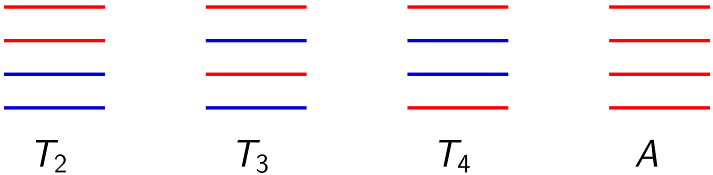

We only are interested in the partition of edges, not in the colours themselves.

"Splitting" of a Fulkerson colouring

"Splitting" of a Fulkerson colouring into two 3-edge-colourings of a 4-edge-cut

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AA

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1 2	1 2
1 2	1 2
1 2	1 3
1 2	1 3
AA	AT_2

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1 2	1 2	1
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1 2	1 3	
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- not all of them are achievable (Kempe chains)

Kempe chains

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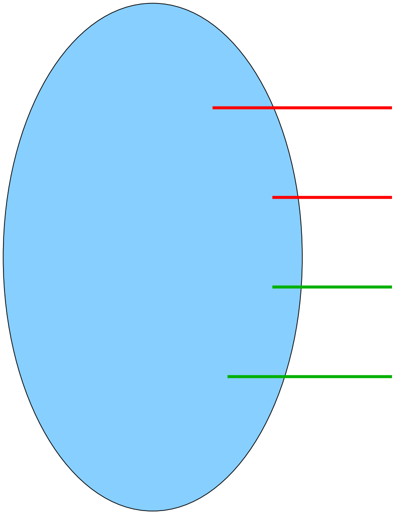

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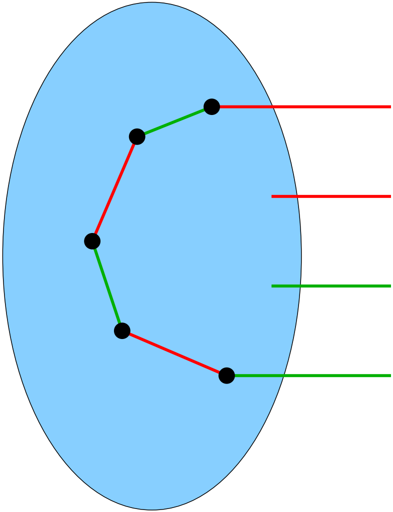

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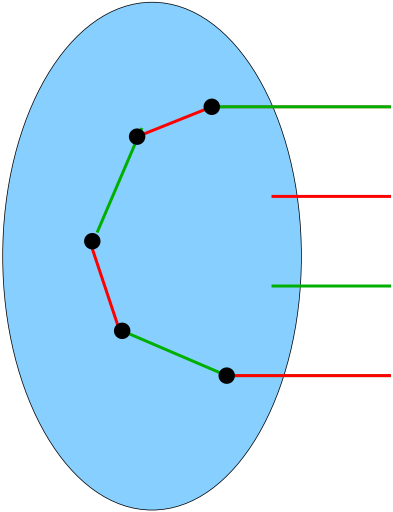
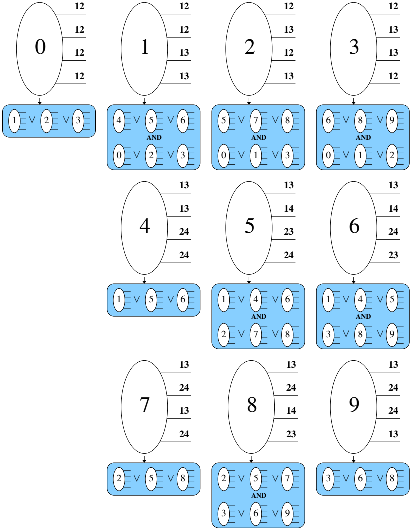
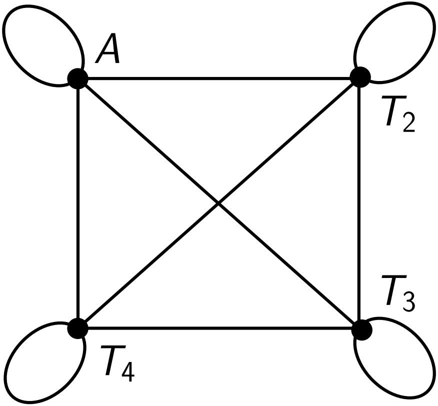


Kempe chains for a Fulkerson colouring



Graph of Fulkerson colourings M

according to a possible Fulkerson colouring, each 4-pole corresponds to a subraph of M



Main result

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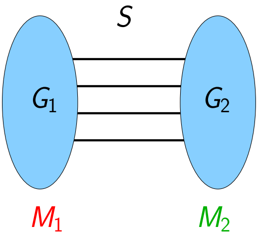

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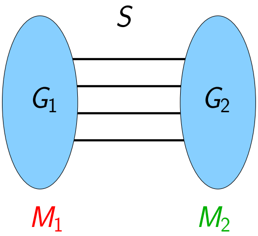

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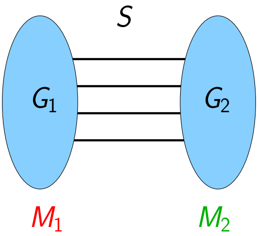

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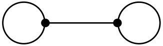
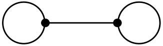
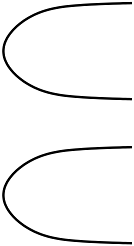
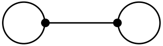
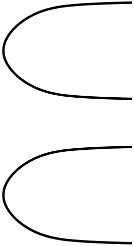
- both M_1 and M_2 are non-empty
- neither M_i nor $\overline{M_i}$ contains a subgraph isomorphic to 

No  subgraph of M_i or $\overline{M_i}$



AA, AT_1, T_1T_1

No  subgraph of M_i or $\overline{M_i}$



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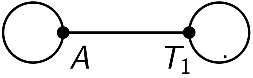
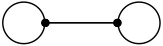
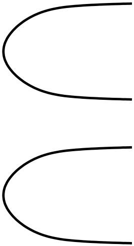


No  subgraph of M_i or $\overline{M_i}$



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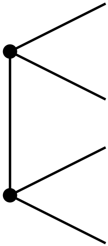
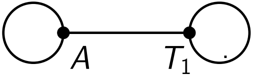
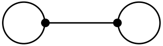
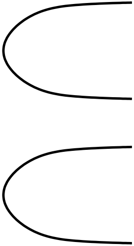
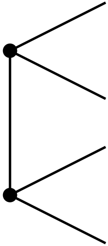


T_3T_3, T_3T_4, T_4T_4



No  subgraph of M_i or $\overline{M_i}$





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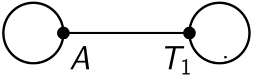
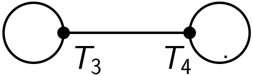
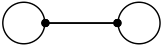
T_3T_3, T_3T_4, T_4T_4





Sketch of the proof

- M_1 and M_2 are edge-disjoint
- both G_1 and G_2 admit a Fulkerson colouring, otherwise we have a contradiction with the minimality of G, therefore
- both M_1 and M_2 are non-empty
- neither M_i nor $\overline{M_i}$ contains a subgraph isomorphic to 

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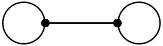
- neither M_i nor $\overline{M_i}$ contains a subgraph isomorphic to 
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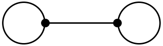
- neither M_i nor $\overline{M_i}$ contains a subgraph isomorphic to 
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- no vertices of degree 2 in M_1 nor M_2 incident with a loop (Kempe chains)

5-edge-cuts

- 56 types of colourings

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- 13 pairs left of sets of colourings

Theorem

Let G be a smallest counterexample to the Fulkerson conjecture. Then G is cyclically 5-edge-connected and every cycle separating 5-edge-cut either separates 5-circuit or separates sets of colourings S_1 and S_2.

Thank you for your attention!